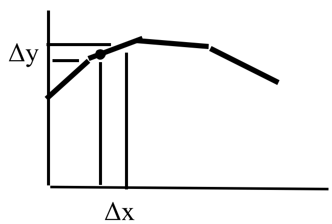
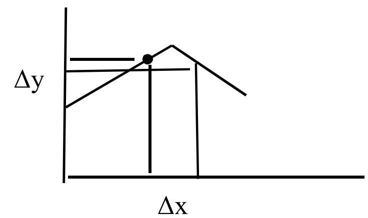
Chapter IV—Differential Calculus

A. Introduction

In differential calculus we look at the ratio of the difference between an incremental change in the value of a function (the “dependent” variable) from some reference as a result of a change in the independent variable. Our interest is the behavior of the function as the result of very small changes in the independent variable, that is, changes where the ratio approaches 0/0. We are concerned about the continuity of the function, which means that as we move along the function towards the reference point, each small change in the position of the independent variable is matched by some small change in the resulting value of the function. Below is an example of a function that is continuous and one that is discontinuous.



continuous



discontinuous

In the continuous function as Δx gets smaller, Δy gets smaller. In the case of this discontinuous function, as Δx gets smaller, Δy at first gets smaller, then gets bigger, and finally gets smaller. That function is discontinuous across the peak. In differential calculus our principal focus is on continuous functions.

B. Derivatives

We can appreciate more fully writing the power of a base number in the form $Ae^{i\theta} = e^{a+i\theta}$ when we explore differential calculus. By knowing how to differentiate this structure we can calculate the derivatives for many of the popular functions. Let us first start with y = a, where “a” is a constant:

$$\begin{aligned} y &= a \\ y + \Delta y &= a \\ \Delta y + y - y &= a - a && \text{Subtracting.} \\ \Delta y &= 0 \\ \Delta y / \Delta x &= 0 \end{aligned}$$

As Δx approaches 0, $dy/dx = 0$. Therefore, $\mathbf{da/dx = 0.}$

Now let us look at

$$\begin{aligned} y &= x: \\ y + \Delta y &= x + \Delta x \\ \Delta y + y - y &= \Delta x + x - x && \text{Subtracting.} \\ \Delta y / \Delta x &= 1 \end{aligned}$$

As Δx approaches 0, $dy / dx = 1$. Therefore, $\mathbf{dx / dx = 1.}$

A very useful formula is:

$$
\begin{aligned}
y &= f(t) \\
y + \Delta y &= f(t + \Delta t) \\
\Delta y &= f(t + \Delta t) - f(t) = \Delta f(t) \\
\Delta y &= (\Delta f(t) / \Delta t)\, \Delta t \\
\Delta y / \Delta x &= (\Delta f(t) / \Delta t)\, (\Delta t / \Delta x)
\end{aligned}
$$

As Δx and Δt approach 0, $dy / dx = (df(t) / dt)\,(dt / dx)$

Therefore, $\mathbf{df(t) / dx = (df(t) / dt)\,(dt / dx)}$

Consider the sum of two functions:

$$
\begin{aligned}
y &= f(u) + g(z) \\
y + \Delta y &= f(u + \Delta x) + g(z + \Delta z) \\
\Delta y &= f(u + \Delta u) - f(u) + (g(z + \Delta z) - g(z)) \\
\Delta f(u) &= f(u + \Delta u) - f(u) \\
\Delta y &= \Delta f(u) + \Delta g(z) \\
\Delta y / \Delta x &= \Delta f(u) / \Delta x + \Delta g(z) / \Delta x
\end{aligned}
$$

As Δx, Δu, and Δz approach 0, $dy / dx = df(u) / dx + dg(z) / dx$.

Thus, $\mathbf{d(f(u) + dg(z)) / dx = df(u) / dx + dg(z) / dx.}$

Consider the product of two functions:

$$
\begin{aligned}
y &= f(u)\, g(z) \\
y + \Delta y &= f(u + \Delta x) g(z + \Delta z) \\
\Delta f(u) &= f(u + \Delta u) - f(u) \\
y + \Delta y &= (f(u) + \Delta f(u))(g(z) + \Delta g(z)) \\
&= f(u)g(z) + \Delta f(u)g(z) + \Delta g(z)f(u) + \Delta f(u)\Delta g(z) \\
\Delta y &= \Delta f(u)g(z) + \Delta g(z)f(u) + \Delta f(u)\Delta g(z) \\
\Delta y / \Delta x &= (\Delta f(u) / \Delta x)\ g(z) + (\Delta g(z) / \Delta x)\ f(u) + (\Delta f(u) / \Delta x)\ \Delta g(z)
\end{aligned}
$$

As Δx, Δu, and Δz approach 0,

$$
\begin{aligned}
\Delta f(u) &= f(u+\Delta u) - f(u) = f(u) - f(u) = 0. \\
\Delta g(z) &= g(z + \Delta z) - g(z) = g(z) - g(z) = 0. \\
dy / dx &= (df(u) / dx)\, g(z) + f(u)(dg(z) / dx) + \Delta g(z)\, (df(u) / dx) \\
dy / dx &= (df(u) / dx)\, g(z) + f(u)\, (dg(z) / dx)
\end{aligned}
$$

Thus, $\mathbf{d(f(u)\, g(z)) / dx = (df(u) / dx)\, g(z) + f(u)\, (dg(z) / dx).}$

We have shown the following equalities:

$$\begin{aligned}
\boldsymbol{d}\mathbf{a} / \mathbf{dx} &= \mathbf{0} \quad \textbf{where a is a constant.}\\
\boldsymbol{d}\mathbf{x} / \mathbf{dx} &= \mathbf{1}\\
\boldsymbol{d}\mathbf{f(t)} / \mathbf{dx} &= \mathbf{(}\boldsymbol{d}\mathbf{f(t)} / \boldsymbol{d}\mathbf{t)(}\ \boldsymbol{d}\mathbf{t} / \boldsymbol{d}\mathbf{x)}\\
\boldsymbol{d}\mathbf{(f(u) + g(z))} / \boldsymbol{d}\mathbf{x} &= \mathbf{(}\boldsymbol{d}\mathbf{f(u)} / \boldsymbol{d}\mathbf{u)(}\ \boldsymbol{d}\mathbf{u} / \boldsymbol{d}\mathbf{x)} + \mathbf{(}\boldsymbol{d}\mathbf{g(z)} / \boldsymbol{d}\mathbf{z)(}\ \boldsymbol{d}\mathbf{z} / \boldsymbol{d}\mathbf{x)}\\
\boldsymbol{d}\mathbf{(f(u)g(z))} / \boldsymbol{d}\mathbf{x} &= \mathbf{(}\boldsymbol{d}\mathbf{f(u)} / \boldsymbol{d}\mathbf{u)(}\ \boldsymbol{d}\mathbf{u} / \boldsymbol{d}\mathbf{x)\ g(z) + f(u)\ (}\boldsymbol{d}\mathbf{g(z)} / \boldsymbol{d}\mathbf{z)(}\ \boldsymbol{d}\mathbf{z} / \boldsymbol{d}\mathbf{x)}
\end{aligned}$$

Let us now look at some popular functions:

a) $y = \log_b(x)$

Let us start by finding the derivative of the inverse of exponentiation. The determination that

$$e = (1 + d)^{1/d} \text{ as d approaches zero}$$

becomes very important in differential calculus. We begin by finding the derivative of ln x.

$$\begin{aligned}
y &= \log_b(x)\\
y + \Delta y &= \log_b(x + \Delta x)\\
y + \Delta y - y &= \log_b(x + \Delta x) - \log_b(x)\\
\Delta y &= \log_b((x + \Delta x)/ x)\\
\Delta y &= ((x\ \Delta x)/ (x\ \Delta x)\ )\ \log_b((x + \Delta x)/ x) \qquad \text{multiply by } 1=(x\ \Delta x)/ (x\ \Delta x)\\
&= (\Delta x / x) \log_b(1 + \Delta x / x)^{(x/\Delta x)}\\
\Delta y / \Delta x &= (1/x) \log_b((1 + \Delta x / x)^{(x/\Delta x)})
\end{aligned}$$

As Δx approaches 0, $(1 + \Delta x / x)^{(x/\Delta x)}$ approaches e.

$$dy / dx = (1/x) \log_b(e)$$

If b = e, then $dy / dx = d \ln (x) / dx = 1 / x$.

We are going to put this in the form: $\boldsymbol{d}\ \mathbf{ln(x)} = \boldsymbol{d}\mathbf{x} / \mathbf{x}$.

b) $y = e^x$

Now that we know that $d \ln(x) = dx/x$, let us find $d(e^x)$.

$$\begin{aligned}
y &= e^x\\
\ln y &= x && \text{Definition of logarithm.}\\
d \ln y &= d\,x\\
dy / y &= dx && \text{Previous derivation.}\\
dy &= y\,dx\\
&= e^x\,dx\\
\boldsymbol{d}\mathbf{e^x} &= \mathbf{e^x}\ \boldsymbol{d}\mathbf{x} && \text{The derivative of } e^x \text{ gives us } e^x.
\end{aligned}$$

c) $y = ax^{bz}$

Now let us go for the most general form: $y = ax^{bz}$.

$$
\begin{aligned}
\ln y &= \ln a + bz \ln x \\
dy / y &= d \ln a + d\,(bz \ln x) \\
&= 0 + (z \ln x)\, d\, b + b\, d(z \ln x) \\
&= 0 + 0 + b\,(z\, d \ln x + \ln x\, dz \\
&= (bz / x)\, dx + b \ln x\, dz \\
dy &= (byz / x)\, dx + by \ln x\, dz \\
\mathbf{dy} &= \mathbf{a\, b(z\, x^{b\,z-1}\, dx + x^{b\,z} \ln x\;\; dz)}
\end{aligned}
$$

This becomes the basic form for the following derivatives:

$d\,a = 0$		$b = 0$
$d\,a\,x = a\,dx$		$b = 1,\ z = 1$
$d\,x^n = nx^{n-1} dx$		$a = b = 1,\ z = n$
$d\,a^x = a^x \ln a\, dx$		$a = b = 1,\ x = a,\ z = x$
$d\,e^x = e^x\, dx$		$a = b = 1,\ x = e,\ z = x$

Here is where we started:

$$
\begin{aligned}
d \ln x &= 1/x \\
d\,x^z &= (\, z\, x^{(z-1)}\, dx + x^z \ln x\;\; dz)
\end{aligned}
$$

Now let us look at the trigonometric functions.

$$
\begin{aligned}
d \sin x &= d(e^{ix} - e^{-ix}) / 2i \\
&= i(e^{ix} + e^{-ix}) / 2i\, dx \\
&= (e^{ix} + e^{-ix}) / 2\, dx \\
&= \cos x\;\; dx
\end{aligned}
$$

$$
\begin{aligned}
d \cos x &= d(e^{ix} + e^{-ix}) / 2 \\
&= i(e^{ix} - e^{-ix}) / 2\, dx \\
&= -\, (e^{ix} - e^{-ix}) / 2i\, dx \\
&= -\sin x\, dx
\end{aligned}
$$

$$
\begin{aligned}
d \tan x &= d \sin x / \cos x \\
&= d\, \sin x\, (\cos x\,)^{-1} \\
&= \sin x\, ((\cos x)^{-2}) \sin x\, dx + (\cos x\,)^{-1} \cos x\, dx \\
&= ((\sin x)^2 / (\cos x)^2 + 1)\, dx \\
&= (((\sin x)^2 + (\cos x)^2\;) / (\cos x)^2\,) dx \\
&= (1 / (\cos x)^2\,)\, dx \\
&= \sec^2 x\, dx
\end{aligned}
$$

Inasmuch as we've been dealing with the inverse functions for the exponential, let us do the same for the trigonometric functions.

$$
\begin{aligned}
y &= \arcsin(x) \\
\sin y &= x \\
d \sin y &= dx \\
\cos y\, dy &= dx \\
dy &= dx / \cos y \qquad \text{but, } (\cos x)^2 + (\sin x)^2 = 1 \\
&= dx / (1 - (\sin y)^2)^{1/2} \quad = dx / (1 - (\sin(\arcsin(x)))^2)^{1/2} \\
&= dx / (1 - x^2)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
y &= \arcsin x = -i \ln (e^{i \arcsin x}) \\
&= -i \ln (\cos (\arcsin x) + i \sin (\arcsin x)) \\
\sin (\arcsin x) &= x \\
\sin^2 z + \cos^2 z &= 1, \text{ so } \cos z = (1 - \sin^2 z)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
y &= \arcsin x \\
&= -i \ln ((1 - x^2)^{1/2} + i x) \\
dy &= d \arcsin x \\
&= (-i (1/2 (1 - x^2)^{-1/2} (-2x) + i) / ((1 - x^2)^{1/2} + ix))\, dx \\
&= (-i (-(1 - x^2)^{-1/2} x + i) ((1 - x^2)^{1/2} - i x) / (1 - x^2 + x^2))\, dx \\
&= (-i (-x + x + i ((1 - x^2)^{1/2} - -(1 - x^2)^{-1/2} x^2)) / 1)\, dx \\
&= -i (i ((1 - x^2)^{1/2} - -(1 - x^2)^{-1/2} x^2))\, dx \\
&= (((1 - x^2) + x^2) / (1 - x^2)^{1/2})\, dx \\
&= 1 / (1 - x^2)^{1/2}\, dx
\end{aligned}
$$

Standard functions written as functions of an exponential and their inverses:

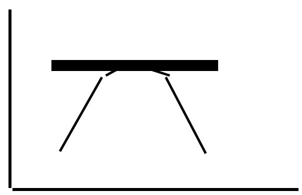
$$
\begin{aligned}
x^y &= e^{y \ln x} \\
\sin x &= (e^{ix} - e^{-ix}) / 2i \\
\arcsin x &= -i \ln ((1 - x^2)^{1/2} + ix)
\end{aligned}
$$

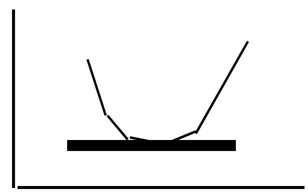
The derivatives of the trigonometric functions are as follows:

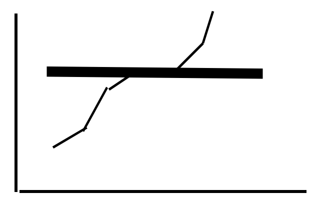
$$
\begin{aligned}
d \sin x &= \cos x\, dx \\
d \cos x &= - \sin x\, dx \\
d \tan x &= (\cos x)^{-2} dx \qquad = \sec^2 x\, dx \\
d \arcsin x &= (1 / (1 - x^2)^{1/2})\, dx
\end{aligned}
$$

C. Maxima and Minima

Once we have familiarized ourselves with taking the derivatives of functions, we can look at some applications. The derivative is defined as $\Delta y/\Delta x$ as Δx approaches zero, so it represents the tangent to the curve at the reference point. At the relative maxima and minima we see that the slope of the tangent is equal to zero. Some curves have an anomolous situation called a saddle point where there is no true maximum or minimum. The following illustrations show curves with, respectively, a maximum, a minimum, and a saddle point.


maximum


minimum


saddle point

Let us consider the curve represented by the equation $y = 2x^2 - 8x + 11 = 2(x-2)^2 + 3$.

$$dy/dx = 4x - 8 = 0, \text{ or } x = 2.$$

Therefore, $x = 2$ for a maximum or minimum.

We know that this curve is a parabola, and from the second form we can tell that the turning point occurs when $x = 2$. The turning point in this case is at $x = 2$, $y = 3$. The minimum value of y is $y = (2)(2^2) - (8)(2) + 11 = 3$.

D. Curve Fitting

Suppose we have a series of points defined by paired coordinates through which we want to draw a line. One definition of the “best” curve to fit all of the data points is that of the “least squares.” We minimize the sum of the square of the distances of each of the points from the line. Here’s how it is calculated:

The equation of the line is $y = ax + b$ The points are x_i, y_i where i goes from 1 to n.

$$s = \sum_{i=1}^{n} (y(x_i) - y_i)^2$$

$$s = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

We choose a and b to minimize this sum.

$$ds/da = 2 \sum_{i=1}^{n.} x_i(ax_i + b - y_i) = 0.$$

$$ds/db = 2 \sum_{i=1}^{n} (ax_i + b - y_i) = 0.$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i x_i = 0.$$

$$a\sum_{i=1}^{n} x_i + nb - \sum_{i=1}^{n} y_i = 0.$$

x	y	xx	xy
1.1	5.1	1.21	5.61
1.9	6.9	3.61	13.11
3.1	9.2	9.61	28.52
3.8	10.0	14.44	38.00
1.3	4.9	1.69	6.37
11.2	36.1	30.56	91.61

Now we merely solve the simultaneous equations for a and b:

$$a = \left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n\sum_{i=1}^{n} x_i y_i\right) / \left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i - n\sum_{i=1}^{n}(x_i)^2\right)$$

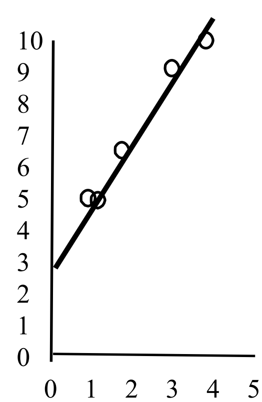
$$b = \left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n}(x_i)^2 \sum_{i=1}^{n} y_i\right) / \left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i - n\sum_{i=1}^{n}(x_i)^2\right)$$

Let us try an example:

$$a = ((11.2)(36.1) - (5)(91.61)) / ((11.2)(11.2) - (5)(30.56))$$
$$b = ((11.2)(91.61) - (30.56)(36.1)) / ((11.2)(11.2) - (5)(30.56))$$

$$a = 1.964$$
$$b = 2.821$$

On the right we have the points and the line that was fitted to them.



E. Power Series

We saw that $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$. We would like to see if we can write other functions as a power series. Let us assume the following:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

Let us take the derivative of f(x)

$$d\,f(x) / dx = \sum_{i=1}^{\infty} i\, a_i x^{i-1}$$

If we let x = 0 we get:

$d(f(x)/dx\,|_o = a_1$ where the notation $|_o$ means "evaluate the left side at x = 0"

and when we take a second derivative, we have:

$$d^2 f(x)/dx^2 = \sum_{i=2}^{\infty} i(i-1)\, a_i\, x^{i-2}$$

If we again let x = 0 we get:

$$d^2 f(x)/dx^2\,|_o = 2!\, a_2$$

If we keep taking derivatives we could conclude that:

$$d^n f(x)/dx^n\,|_o = n!\, a_n$$

and

$$f(x) = \sum_{i=0}^{\infty} d^{\,i} f(x)/dx^i\,|_o\, x^i/i!$$

Let us try this with $y = \tan x$

$$\begin{aligned}
dy/dx &= \cos^{-2}(x) \\
d^2y/dx^2 &= 2\sin(x)\cos^{-3}(x) \\
d^3y/dx^3 &= 2\cos^{-2}(x) + 6\sin^2(x)\cos^{-4}(x) = -4\cos^{-2}(x) + 6\cos^{-4}(x) \\
d^4y/dx^4 &= -8\sin(x)\cos^{-3}(x) + 24\sin(x)\cos^{-5}(x) \\
d^5/dx^5 &= -8\cos^{-2}(x) - 24\sin^2(x)\cos^{-4}(x) + 24\cos^{-4}(x) + 120\sin^2(x)\cos^{-6}(x) \\
y|_0 &= 0 \\
dy/dx|_0 &= 1 \\
d^2y/dx^2|_0 &= 0 \\
d^3y/dx^3|_0 &= -4 + 6 = 2 \\
d^4y/dx^4|_0 &= 0 + 0 + 0 \\
d^5/dx^5|_0 &= -8 + 24 = 16 \\
\tan x &= f(x)|_o + df(x)/dx\,|_o x + d^2 f(x)/dx^2\,|_o x^2/2! + d^3 f(x)/dx^3|_o x^3/3! + \ldots \\
&= x + 2x^3/3! + 16x^5/5! + \ldots \\
&= x + x^3/3 + 2x^5/15 + \ldots
\end{aligned}$$

We can verify this by dividing the power series for the sine and cosine functions.

F. Summary

Function	Derivative
x	1
yz	z dy/dx + y dx/dx
ay^{bz}	$a\,b(z\,y^{b\,z-1}\,dy/dx + y^{b\,z}\ln y\;dz/dx)$
ln(x)	1 / x
sin(x)	cos(x)
cos(x)	- sin(x)
$\sin^{-1}(x)$	$1 / (1 - x^2)^{1/2}$

G. Practice

1. Find dy/dx when $y = \ln(\cos^3(\sin^{-1}(x)))$.
2. Find the maxima and minima for the following functions:
   a) $x^2 + y^2 = r^2$
   b) $A = 2\pi r^2 + 2\pi rh$, where r and h are constants.
3. Find a power series for $\tan^{-1}(x)$.
4. Find $\tan^{-1}(\sqrt{3} / 3)$.