# Chapter V—Integral Calculus

## A. Introduction

Whereas differential calculus is concerned with the ratio of two quantities as changes in them approach zero, integral calculus focuses on the summation of small quantities as the number of quantities becomes infinitely large. There is a geometric interpretation to the calculus: if you plot a continuous function as a curve on a Cartesian coordinate graph, the differential would be the tangent to the curve at the point of selection, while the integral is interpreted as the area between the curve and the x-axis.

We used the logarithm to build up the base for the differential calculus. Now we'll use the exponential as our base for integration. When I was taught integration I never fully understood that integration could be used to define functions (although not every integration can be translated into our small set of commonly used trigonometric and polynomial functions). Once I realized that the trigonometric functions could be represented by exponentials, and that I could define functions, I improved my understanding of number structure and could create tables for new and old functions.

## B. Integrations

Let us look at the integration of $e^x$.

$$\int_a^x e^{bx}\,dx = \lim_{\Delta x \to 0} \sum_{m=0}^{n} e^{(bm\,\Delta x + ba)}\,\Delta x \qquad \text{Summation of a geometric series.}$$

$$= \lim_{\Delta x \to 0} \Delta x\, e^{ba}(e^{((n+1)\,\Delta x)} - 1) / (e^{b\Delta x} - 1) = (e^{bx} - e^{ba}) / b$$

where $n\Delta x + a = x$

Note:

$$\begin{aligned}(e^{b\Delta x} - 1) / \Delta x &= (1 + b\Delta x + (b\Delta x)^2 / 2! + (b\Delta x)^3 / 3! + \ldots - 1) / \Delta x \\ &= b + b(b\Delta x) / 2! + b(b\Delta x)^2 / 3! + \ldots \\ &= b \text{ as } \Delta x \to 0.\end{aligned}$$

$$\boldsymbol{\int_a^x e^{bx}\,dx = e^{bx} / b - e^{ba} / b}$$

A simpler integration is that of b: $\boldsymbol{\int b\,dx}$

$$\int_a^x b\,dx \;=\; \lim_{\Delta x \to 0} \; b \sum_{m=0}^{n} \Delta x \;=\; \lim_{\Delta x \to 0} bn\Delta x \;=\; b(x - a)$$

$$\int_a^x \mathbf{b\,dx} \;=\; \mathbf{b(x - a)}$$

## C. Anti-differentiation

Let us look at the integral of a differentiated function.

$$\int_a^x d\,f(x)/dx\;dx \;=\; \sum_{m=1}^{n} \left(f(a + m\Delta x) - f(a + (m - 1)\Delta x)\right) / \Delta x)\;\; \Delta x$$

$$= \sum_{m=1}^{n} (f(a + m\Delta x) - f(a + (m - 1)\Delta x)$$

$$= f(a + n\Delta x) - f(a)$$

$$= f(x) - f(a)$$

By leaving out the limits of the integral, we introduce the concept of an indefinite integral. For example, $\int d\,f(x) / dx\;dx \;=\; f(x) + c$, where $c \;=\; -f(a)$.

Now let us look at the integration of a product. In differential calculus we discovered that the derivative of a product was as follows:

$$\boldsymbol{d(f(u)g(z)) / dx \;=\; (df(u) / dx)\, g(z) + f(u)\,(dg(z) / dx)}$$

Let us integrate this expression:

$$\int d(f(u)\,(g(z)) / dx)\,dx \;=\; \int (df(u) / dx)\,g(z)\,dx + \int f(u)\,(dg(z) / dx)\,dx$$
$$\int f(u)\,(dg(z) / dx)\,dx \;=\; \int d(f(u)\,g(z)) / dx)dx - \int (df(u) / dx)\,g(z)\,dx$$
$$=\; (f(u)\,g(z)) - \int (df(u) / dx)\,g(z)\,dx$$

There is an implied constant that we can evaluate when we know the limits: $\boldsymbol{\int x^a dx}$

Let us try the product formula on the following:

$$\int x^a\,dx = x^a x - \int x\,dx^a$$

$$= x^{a+1} - \int a x x^{a-1} dx$$

$$= x^{a+1} - \int a x^a dx$$

$$(a+1)\int x^a dx = x^{a+1} \qquad \text{Adding } \int a x^a dx \text{ to } \int x^a dx.$$

$$\boldsymbol{\int x^a dx = x^{a+1}/(a+1)} \qquad \text{Dividing by } a + 1.$$

## D. Logarithms Revisited

Logarithms play a very important part in solving a whole host of integration problems. We only know that taking the logarithm is the inverse of exponentiation, but we do not yet have a power series with which to determine numerical values. If we tried to evaluate

$$\int x^a\,dx = x^{a+1}/(a+1)$$

for a = -1, we'd get:

$$\int x^{-1}\,dx = x^{-1+1}/(-1+1)$$

$$= 1/0, \qquad \text{an indeterminate answer.}$$

Let us define the ln(y) as follows:

$$\ln(y) = \int (1/y)\,dy$$

And now let us substitute $y^a$ for y.

$$\ln y^a = \int 1/y^a\,dy^a = \int (a y^{a-1})/y\,dy = a\int 1/y\,dy = a \ln y$$

This result is the same as the logarithm of an exponential.

Let us try the logarithm of a product:

$$\ln(xy) = \ln(x) + \ln(y) = \int dx/x + dy/y = \int (y\,dx + x\,dy)/xy = \int d(xy)/xy$$

Let us try $e^y$

$$\ln(e^y) = \int 1/e^y\,de^y = \int e^y/e^y\,dy = \int dy = y$$

Thus, ln y is the inverse of $e^y$.

The logarithmic properties of exponentiation and multiplication are satisfied by this function. We have used integration to define a function, as contrasted to defining a function of multiple operations upon a known set of functions. Let us see if we can develop a power series for this function.

$$\ln(1 + y) = \int d(\ln(1 + y)) = \int dy / (1 + y)$$

$$\int dy / (1 + y) = \int (1 - y + y^2 - y^3 + y^4 + \ldots)dy \qquad (1 / (1 + y) = 1 - y + y^2 - y^3 + \ldots)$$

$$= y - y^2 / 2 + y^3 / 3 - y^4 / 4 + \ldots$$

Substituting x for 1 + y:

$$\mathbf{\ln x = (x - 1) - (x - 1)^2 / 2 + (x - 1)^3 / 3 - (x - 1)^4 / 4 + \ldots \text{ for } 1 < x < 2.}$$

To evaluate this expresion we substitute y for –y and subtract the one equation from the other.

$$\ln((1 + y) / (1 - y)) = \ln(1 + y) - \ln(1 - y) = 2(y + y^3 / 3 + y^5 / 5 + \ldots )$$

Now let us make some substitutions so that we can calculate the values of some logarithms:

$$y = 1 / 5 \qquad \ln ((6 / 5) / (4 / 5) ) = \ln (3 / 2) = \ln 3 - \ln 2 = a$$
$$y = 1 / 7 \qquad \ln ((8 / 7) / (6 / 7) ) = \ln (4 / 3) = 2 \ln 2 - \ln 3 = b$$
$$\ln 2 = a + b$$
$$\ln 3 = 2a + b$$
$$a = 2 (1 / 5 + 1 / 1875 + 1 / 15625) = .40546510$$
$$b = 2 (1 / 7 + 1 / 1029 + 1 / 84035) = .28768207$$
$$\ln 2 = .69314718053$$
$$\ln 3 = 1.09861222863$$
$$\ln 4 = \ln (2 \times 2) = \ln 2 + \ln 2 = 2 \ln 2$$
$$\ln 5 = 1.60943785238$$
$$\ln 6 = \ln (2 \times 3) = \ln 2 + \ln 3$$
$$\ln 7 = 1.94591014\text{xxxx}$$
$$\ln 10 = \ln (2 \times 5) = \ln 2 + \ln 5 = 2.30258503292$$

Suppose we were interested in calculating the logarithms of the prime numbers. Using the ratio

$$(1 + x) / (1 - x)$$

we try different values for x that will allow us to calculate the logarithms of the prime numbers up to 19.

| | x | (1+x)/(1-x) | | |
|---|---|---|---|---|
| | 1 / 7 | 8 / 6 | = 4 / 3 | = 2 x 2 / 3 |
| a | 1 / 17 | 18 / 16 | = 9 / 8 | = 3 x 3 / 2 x 2 x 2 |
| b | 1 / 19 | 20 / 18 | = 10 / 9 | = 5 x 2 / 3 x 3 |
| | 1 / 23 | 24 / 22 | = 12 / 11 | = 2 x 2 x 3 / 11 |
| | 1 / 25 | 26 / 24 | = 13 / 12 | = 13 / 2 x 2 x 3 |
| | 1 / 27 | 28 / 26 | = 14 / 13 | = 2 x 7 / 13 |
| | 1 / 29 | 30 / 28 | = 15 / 14 | = 3 x 5 / 2 x 7 |
| c | 1 / 31 | 32 / 30 | = 16 / 15 | = 2 x 2 x 2 x 2 / 3 x 5 |
| | 1 / 33 | 34 / 32 | = 17 / 16 | = 17 / 2 x 2 x 2 x 2 |
| | 1 / 37 | 38 / 36 | = 19 / 18 | = 19 / 2 x 3 x 3 |

$a = 2 \ln 3 - 3 \ln 2$
$b = \ln 5 + \ln 2 - 2 \ln 3$
$c = 4 \ln 2 - \ln 3 - \ln 5$

$\ln 2 = 3a + 2(b + c)$
$\ln 3 = 5a + 3(b + c)$
$\ln 5 = 7a + 5b + 4c$

The logarithms (to the base "e") of the number from 1 to 10 are as follows:

$\ln 1 = 0$
$\ln 2 = .69314718053$
$\ln 3 = 1.09861222863$
$\ln 4 = 2 \ln 2$
$\ln 5 = 1.60943785238$
$\ln 6 = \ln 2 + \ln 3$
$\ln 7 = 1.94591014$xxxx
$\ln 8 = 3 \ln 2$
$\ln 9 = 2 \ln 3$
$\ln 10 = \ln 2 + \log 5 = 2.30258503292$

$\text{Log } 2 = .69314718053 / 2.30258503292 = .30103.$

In other words, integration allows us to define the properties of a function and gives us a way to calculate numerical values. The functions $e^x$ and ln x are inverses of one another, just as differentiation and integration are inverse functional operations of one another. It was through a differentiation process that we could determine the value of "e" and through an integration process that we could evaluate the logarithmic function.

Let us see if the power series solution works with complex numbers.

$$\ln(1+y) = \int(1/(1+y))\,dy = \int(1/y - 1/y^2 + 1/y^3 - 1/y^4 + )dy$$

$$\ln(1+y) - \ln y = \ln(1 + 1/y) = 1/y - 1/2y^2 + 1/3y^3 - 1/4y^4 + 1/5y^5$$

Substituting x for 1 / y we get the same series for the ln as with the other approach:

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4$$

Let $x = i / 3^{1/2}$. Then

$$\ln(1 + i/3^{1/2}) = i(3^{1/2}/2)\,(1 - 1/(3 \times 3^1) + 1/(5 \times 3^2) - 1/(7 \times 3^3) + 1/(9 \times 3^4) + \ldots)$$
$$+ 1/(2 \times 3) - 1/(4 \times 3^2) + 1/(6 \times 3^3) - 1/(8 \times 3^4) + 1/(10 \times 3^5) + \ldots$$

Since $i\pi / 6 \quad = \ln(3^{1/2}/2 + i/2) = \ln(1 + i/3^{1/2}) + \ln(3^{1/2}/2)$

then $\pi/6 \quad = (3^{1/2}/2)\,(1 - 1/(3 \times 3^1) + 1/(5 \times 3^2) - 1/(7 \times 3^3) + 1/(9 \times 3^4) + \ldots)$

and $\ln 2 - (\ln 3)/2 \quad = (1/(2 \times 3) - 1/(4 \times 3^2) + 1/(6 \times 3^3) - 1/(8 \times 3^4) + 1/(10 \times 3^5) + \ldots)$.

Performing the calculations validates the formulations. This exercise was done to convince us that the expansion series obeys the same rules as the function it defines.

## E. Calculating π

Let us use integration to evaluate π more accurately than we're previously done:

$$\begin{aligned} \arcsin x &= \int(1 - x^2)^{-1/2}\,dx \\ &= \int(1 + x^2/2 + 3x^4/2^3 + 5x^6/2^4 + 5 \times 7\, x^8/2^7 + \ldots)dx \\ &= x(1 + (2/3)(x/2)^2 + (2 \times 3/5)(x/2)^4 + (4 \times 5/7)(x/2)^6 + (5 \times 7/9)(x/2)^8 + \ldots \end{aligned}$$

$$\begin{aligned} \arcsin 1/2 &= \pi/6 \\ \pi &= 3(1 + 1/(3 \times 2^3) + 3/(5 \times 2^7) + 5/(7 \times 2^{10}) + 5 \times 7/9\,(2^{15}) + \ldots \\ &= 3.1415 \end{aligned}$$

This gives us a faster approach to calculating pi than the sum of the reciprocal squares.

## F. Integration Techniques

We do not want arbitrarily to define a new function when we have difficulty performing an integration. There are several techniques we could employ. These include partial fractions, trigonometric substitutions, and "integration by parts."

Consider the following integral:

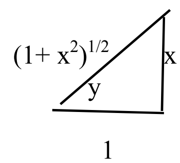
$$\int dx / (1 + x^2)$$

Our first approach is to solve it by partial fractions:

$$\begin{aligned} \int dx / (1 + x^2) &= \tfrac{1}{2} \int (1 / (1 - xi) + 1 / (1 + xi))dx \\ &= -i / 2\, (\ln(1 + xi) - \ln(1 - xi)) \\ &= i / 2\, \ln((1 - xi) / (1 + xi)) \end{aligned}$$

If $\sin(y) = x / (1 + x^2)^{1/2}$, then $\tan(y) = x$.

$$\begin{aligned} (i/2) \ln((1 - xi)/(1 + xi)) &= (i/2)\, (\ln(e^{-i \arctan(x)}) - \ln( e^{i \arctan(x)})) \\ &= \arctan(x) \\ \int dx / (1 + x^2) &= \arctan(x) \end{aligned}$$



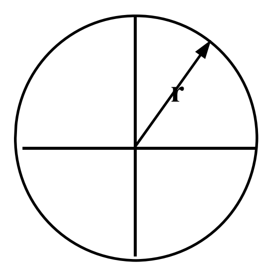

Let us use a trigonometric substitution:

$$\begin{aligned} x &= \tan(y) \\ dx &= dy / \cos^2(y) \\ \int dx / (1 + x^2) &= dy /((\cos^2(y))(1 + \tan^2(y))) = \int dy \\ &= y \\ &= \arctan(x) \end{aligned}$$

## G. Calculating Length

The length of an arc is the sum of an infinite number of infinitesimal lengths:

$$\begin{aligned} L &= \Sigma\, (\Delta x^2 + \Delta y^2)^{1/2} \\ &= \int (1 + (dy/dx)^2)^{1/2}\, dx. \end{aligned}$$




Let us find the length of a semicircle. The equation of a circle is:

$$\begin{aligned} y &= (r^2 - x^2)^{1/2}. \\ dy/dx &= -x / (r^2 - x^2)^{1/2} \end{aligned}$$

Then the length is:

$$\begin{aligned} \int_{-r}^{r} (1 + x^2 / (r^2 - x^2))^{1/2} dx &= r\int_{-r}^{r} dx / (r^2 - x^2)^{1/2} \\ &= r(\sin^{-1}(x / r))\Big|_{-r}^{r} \\ &= r(\pi / 2 - -\pi / 2) \\ &= \pi r \end{aligned}$$

## H. Summary

$$\int_a^x e^{bx}\,dx = e^{bx}/b - e^{ba}/b$$

$$\int_a^x b\,dx = b(x - a)$$

$$\int_0^x x^a\,dx = x^{a+1}/(a + 1)$$

$$\int_1^x dx/x = \ln(x)$$

The length of an arc = $\int(1 + (dy/dx)^2)^{1/2}\,dx$.

## I. Practice

1. Show that $\int x\,(x + 1)^{1/2}\,dx = ((6x - 4)\,(x + 1)^{3/2}) / 15$.

2. Show $\int \ln(x)dx = x\,(\ln(x) - 1)$.

3. Show $\int \tan x = -\ln(\cos(x))$.

4. Show $\int \sec(x)\,dx = \ln(\sec(x) + \tan(x))$.

5. Show $\int \arcsin(x)\,dx = x \arcsin(x) + (1 - x^2)^{1/2}$.

6. Show $\int(1 - \cos(x))^{1/2}\,dx = -2\sqrt{2}\cos(x/2)$.

7. Show $\int_0^\infty x^{-t}e^{x}dx = t\int_0^\infty x^{-t-1}e^{x}\,dx$