Generating Trigonometric Tables

We generate the trigonometric tables by using the trigonometric formula and the powers series for the sine and cosine.

$\sin^2(x)+\cos^2(x)=1$
$\sin(-x)=-\sin(x)$
$\cos(-x)=\cos(x)$
$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$
$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$
$\sin(0)=0$
We derive
$\sin(x)=\cos(90-x)$
$\sin(2x)=2\sin(x)\cos(x)$
$\cos(x)=((1+\cos(2x))/2)^{\wedge}.5$

$\cos(90)=0$
$\cos(45)=((1+0)/2)^{\wedge}.5=(\sqrt{2})/2$

$\sin(30)=\cos(60)$
$0=\cos(90)=\cos(60+30)=\cos(60)\cos(30)-\sin(30)\sin(60)=\sin(30)\cos(30)-2\sin^2(30)\cos(30)$
$0=\sin(30)\cos(30)(1-2\sin(30))$
$\sin(30)=1/2=\cos(30)$
$\text{Cos}(60)=\sin(30)=(1-\cos^2(30))^{1/2}=3^{1/2}/2$

$\text{Sin}(36)=\cos(54)=\cos(36+18)=\cos(36)\cos(18)-\sin(36)\sin(18)$
$\sin(36)(1+\sin(18))=\cos(36)\cos(18) =(1-2\sin^2(18))\cos(18)$
$2\sin(18)\cos(18)(1+\sin(18)=(1-2\sin^2(18))\cos(18)$
$\cos(18)(4\sin^2(18)+2\sin(18)-1)=0$
$\sin(18)=(-2+(4+16)^{1/2})/8=(5^{1/2}-1)/4=\cos(72)$
$\text{Cos}(36)=((1+\cos(72))/2)^{1/2}=(1+(5^{1/2}-1)/4)/2)^{1/2}=(5^{1/2}+1)/4$
$\text{Cos}(75)=\cos(45)\cos(30)-\sin(45)\sin(30)=2^{1/2}(3^{1/2}-1)/4$

Let us build our table

x	formula	Cos(x)
90		0
81	cos(9)cos(72)/(1+cos(18))	.1564
75		.2588
72		.3090
63	cos(27)cos(36)/(1+cos(54))	
60		.5
54	$\sin(36)=(1-\cos^2(36))^{1/2}$	.5878
45		.7071
36	$((1+\cos(72)/2)^{1/2}$	
30		.8660
27	$((1+\cos(54)/2)^{1/2}$	.8910
18	$((1+\cos(36))/2)^{1/2}$	.9510
15	$((1+\cos(30))/2)^{1/2}$	.9659
9	$((1+\cos(36))/2)^{1/2}$	.9876
0		1

Where we have no formula, we have shown how to calculate the formula or value on the previous page. Experience shows us that we want to avoid subtractions if possible in numeric calculations because it takes more effort to get an accurate answer. Note that since the sin(x) = cos(90-x), we convert all of our sines to cosines so that we can make use of previous calculations. To get the items in the table we find the vales in the following order. Since we know th cosine of 72, half angle calculations give us 36, 18, and 9. From 30 we get 15. For 54, we did the following: sin(36)=cos(90-36)=cos(54). The $\sin^2(36)+\cos^2(36)=1$. Then we used the half angle formula for 27.

For 81 and 63, we used a little algebra

Cos(81)=cos(9+72)=cos(9)cos(72)-sin(9)sin(72)
=cos(9)cos(72)-cos(81)sin(72)
Cos(81)(1+sin(72)=cos(9)cos(72)   sin(72)=cos(18)
cos(81)=cos(9)cos(72)/(1+cos(18)

Similarly cos(63)=cos(27+36)
cos(63)=cos(27)cos(36)/(1+cos(54))

We find cos(84)=cos(75+9), cos(42) as a half angle of 84 and cos(48) = sin(90-48) = $(1-\cos^2(42))^{1/2}$. You can now figure out the patterns for the other angles.

To calculate the cos(1) and sin(1), we use the following formula:

$\cos(x+x0)=\cos(x0)\cos(x)-\sin(x0)\sin(x)$
$=(1-x^2/2!+x^4/4!-x^6/6!+\ )\cos(x0)-(x-x^3/3!+x^5/5!-\ \ldots)\sin(x0)$
$\sin(x+x0)=\sin(x0)\cos(x)+\sin(x)\cos(x0)$
$+(1-x^2/2!+x^4/4!-x^6/6!+\ )\sin(x0)+(x-x^3/3!+x^5/5!-\ \ldots)\cos(x0)$
Let x0=0 and $x=pi/180=1^0$
$\cos(1)=(1-x^2/2!+x^4/4!-x^6/6!+\ )$

We need to calculate pi.
We develop the Machin formula.
tan(a)= 1/5 or a=atan(1/5)
tan(x+y)=(tan(x)+tan(y))/(1-tan(x)tan(y))
$\tan(2a)= 2\tan(a)/(1-\tan^2(a))= (2/5)/(1-1/25)=10/24=5/12$
$\tan(4a)=2\tan(4a)/(1-\tan^2(4a))=(10/12)/(1-25/144)=120/119$
tan(pi/4-4a)=(tan(pi/4)-tan(4a))/(1+tan(pi/4)tan(4a))=(1-tan(4a))/(1+tan(4a))=
(1-120/119)/(1+120/119)=-1/239
pi/4-4a=atan(1/239)
pi=16atan(1/5)-atan(1/239)

From calculus
$atan(x)=x-x^3/3+x^5/5-x^7/7+$
atan(1/5)=1/5-1/375+1/15625-1/546875=.1973955
atan(1/239)=1/239-1/40955757=.00418407
pi=3.15832807-0.01673628=3.141592
pi/180=.017453

We could have evaluated the sines for one degree angles by using the sin of a triple angle formula and used the Newton-Rapheson approach for solving the equation. In this example, we are finding the sine(10), which could be used with the sin(9) and cos(9) to find the cosine(1).

$\sin(2x)=2\sin(x)\cos(x) \quad \cos(2x)=\cos^2(x)-\sin^2(x)=1-2\sin^2(x)$
$\sin(3x)=\sin(x+2x)=\sin(x)\cos(2x)+\cos(x)\sin(2x)=\sin(x)(1-2\sin^2(x)+2\cos(x)\sin(x)\cos(x)$
$=\sin(x)(1-2\sin^2(x))+2\sin(x)\cos^2(x)=\sin(x)(1-2\sin^2(x)+2-2\sin^2(x))$

$4\sin^3(x)-3\sin(x)+\sin(3x)=0$
Let 3x=30
$z=4\sin^3(x)-3\sin(x)+1/2=0 \quad y=\sin(x)$
$z=8y^3-6y+1=0$
$z'=24y^2-6$
$y=y-z'/y=(16y^3-1)/(24y^2-6)$ a recursive formula
Let y=.25 $y=(16x.25^3-1)/(24x.25^2-6)= -.75/(-4.5)=.1666\ldots$

y=.16	y=.1735
y=.1735	y=.17364816
y=.17364816	y=0.173648177666930

The next try we would get the sin(10) to 30 decimal places.
By deriving the values for the trigonometric tables, we have seen why we had to learn how to derive the trigonometric formula. Even though we have not learned calculus, we can see why the knowledge of calculus was necessary to extend those formula.

As we have learned to derive the logarithmic and trigonometric tables, it became obvious why we had to learn algebra, trigonometry, and differential and integral calculus.