# Chapter VI—Trigonometry

## A. Introduction

Trigonometry traditionally sets a template for studying a small set of related functions. We try to maintain some of that philosophy here, but trigonometry does more than that. It helps us complete the foundations for number structure and it introduces us to the concept of a periodic function. We also make use of this number structure link to prove trigonometric relationships without geometric construction. This should set a person to wonder how we can link an abstract mathematical relationship to a pictorial geometric relationship.

## B. Definition

Trigonometry is a branch of mathematics strongly focused on the study and measure of angles. Though the name translates to the study of triangles, triangles are only a starting point, because any polygon can be constructed as a set of triangles. Any closed geometric figure can be approximated by a many sided polygon, where as the number of sides increases without limit, the polygon becomes a better and better approximation of the figure—a calculus of geometric shapes. Trigonometry is a specialized link between the structure of numbers and geometry.

We can use the real numbers to measure sides and areas. From the complex numbers we get a measure of angles. Is this an accident or is there some simple theory that can explain the observed physical phenomenon about us? A person studying advanced fields of science begins to learn about scientific discoveries that were predicted by mathematics, or how once something was discovered, a field of mathematics was applied to understanding this discovery. Is it possible that because we’ve boxed in our thinking with the learning of mathematics that we have delayed some monumental discoveries?

Trigonometry and number structure are linked by the Pythagorean theorem, in which the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. It is this step in pattern recognition that allows us to take what we have learned in number structure and apply it easily to trigonometry.

During our study of the structure of numbers we introduced the functions *sine* and *cosine*. They were the real and imaginary parts, respectively, of $e^{i\theta}$. By squaring the polynomial series that represents these numbers and then adding them together, we showed the relationship

$$\sin^2\theta + \cos^2\theta = 1.$$

Then using the rules for operating with exponents, we developed the trignometric formulae for sines and cosines. We showed that the formulae suggest the functions are periodic. Let us review that process:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\begin{aligned}
\cos(x + y) + i\sin(x + y) &= e^{i(x+y)} \\
&= e^{ix}e^{iy} \\
&= (\cos x + i\sin x)(\cos y + i\sin y) \\
&= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y)
\end{aligned}$$

Thus, $\cos(x + y) = \cos x\cos y - \sin x\sin y$ Equating the real parts. (1)
and $\sin(x + y) = \sin x\cos y + \cos x\sin y$ Equating the imaginary parts. (2)
Also,

$$\begin{aligned}
\cos(x - y) + i\sin(x - y) &= e^{i(x-y)} \\
&= e^{ix}e^{-iy} \\
&= (\cos x + i\sin x)(\cos y - i\sin y) \\
&= (\cos x\cos y + \sin x\sin y) + i(\sin x\cos y - \cos x\sin y)
\end{aligned}$$

Thus, $\cos(x - y) = \cos x\cos y + \sin x\sin y$ Equating the real parts.
and $\sin(x - y) = \sin x\cos y - \cos x\sin y$ Equating the imaginary parts.

If we let y = x in equation (1), we have:

$$\begin{aligned}
\cos(2x) &= \cos x\cos x - \sin x\sin x \\
&= \cos^2\theta - \sin^2\theta \\
&= \cos^2\theta - (1 - \cos^2\theta) \\
&= 2\cos^2\theta - 1.
\end{aligned}$$

From the above formula and the relationship $\sin^2\theta + \cos^2\theta = 1$ we can see that there is a quarter period parameter p such that $\sin p = 1$ and $\cos p = 0$.

Then $\cos(2p) = 2\cos^2 p - 1 = -1$ and $\sin(2p) = 0$.

Using equations (1) and (2) we have

$$\begin{aligned}
\cos(3p) &= [\cos(2p)][\cos(p)] - [\sin(2p)][\sin(p)] \\
&= (-1) \times 0 - 0 \times 1 \\
&= 0 \\
\sin(3p) &= [\sin(2p)][\cos(p)] + [\cos(2p)][\sin(p)] \\
&= 0 \times 1 + (-1) \times 1 \\
&= -1
\end{aligned}$$

and

$$\begin{aligned}
\cos(4p) &= [\cos(2p)][\cos(2p)] - [\sin(2p)][\sin(2p)] \\
&= (-1) \times (-1) - 0 \times 0 \\
&= 1 \\
\sin(4p) &= [\sin(2p)][\cos(2p)] + [\cos(2p)][\sin(2p)] \\
&= 0 \times (-1) + (-1) \times 0 \\
&= 0.
\end{aligned}$$

Putting this in a table, we have:

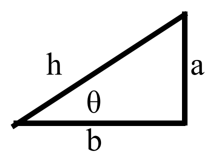
| p | sin p | cos p |
|---|---|---|
| 0 | 0 | 1 |
| p | 1 | 0 |
| 2p | 0 | -1 |
| 3p | -1 | 0 |
| 4p | 0 | 1 |

By assumption and the relation $\sin^2\theta + \cos^2\theta = 1$.

The cycle repeats at 4p. We determined that $p = \pi/2$, so the cycle is $2\pi$.

In trigonometery we define the sine of an angle as the ratio of the side opposite the hypotenuse to the hypotenuse in a right triangle. We define the cosine of the angle as the side adjacent to the hypotenuse to the hypotenuse in a right triangle. From the following diagram, we have:

$$\sin\ \theta = a / h$$
$$\cos\ \theta = b / h$$




$$\begin{aligned} \sin^2\theta + \cos^2\theta &= (a / h)^2 + (b / h)^2 \\ &= (a^2 + b^2) / h^2 \\ &= h^2 / h^2 \\ &= 1 \end{aligned}$$

When $a = 0$, $\sin\theta = 0$ and $\cos\theta = 1$.

Through geometric construction we can derive the sine and cosine of the sum of two angles. With close observation it becomes apparent that the Pythagorean theorem provided the key link to drawing the parallel between the variable in $e^{i\theta}$ and the $\theta$ in the geometric figure.

It is useful in applications of trigonometry to define the tangent of the angle as the ratio of the sine to the cosine. $\tan\theta = \sin\theta / \cos\theta = a / b$ in our example. We can also give names to the reciprocals of the sine, cosine, and tangent, respectively, as the cosecant, secant, and cotangent.

We can develop interesting new relationships using $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta / \cos\theta$:

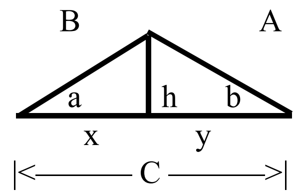
$$\begin{aligned} \tan(x + y) &= \sin(x + y) / \cos(x + y) \\ &= ([\sin(x)][\cos(y)] + [\sin(y)][\cos(x)]) / ([\cos(x)][\cos(y)] - [\sin(x)][\sin(y)]). \end{aligned}$$

Dividing by $[\cos(x)][\cos(y)]$,

$$\tan(x + y) = (\tan(x) + \tan(y)) / (1 - [\tan(x)][\tan(y)]).$$

## C. Triangle Formulae

With one side and any other two of the other five parts (angles and sides) of any triangle, we can determine the remaining three parts.




$$
\begin{aligned}
\sin(a) &= h / B, \quad \sin(b) = h / A \\
\sin(a) / \sin(b) &= A / B, \text{ or} \\
A / \sin(a) &= B / \sin(b)
\end{aligned}
$$

$$\cos(a) = x / B, \quad \cos(b) = y / A$$

$$
\begin{aligned}
\sin(c) &= \sin(90 - b + 90 - a) \\
&= \sin(a + b) \\
&= [\sin(a)][\cos(b)] + [\sin(b)][\cos(a)] \\
&= (h / B)(y / A) + (h / A)(x / B) \\
&= (x + y)\, h / (AB) \\
&= Ch / AB \\
&= (C / B)[\sin(b)]
\end{aligned}
$$

$$
\begin{aligned}
\sin(c) / \sin(b) &= C / B \\
B / \sin(b) &= C / \sin(c)
\end{aligned}
$$

This gives us the **law of sines: $A / \sin(a) = B / \sin(b) = C / \sin(c)$**

Using the figure again:

$$
\begin{aligned}
x &= (B^2 - h^2)^{1/2} \\
y &= (A^2 - h^2)^{1/2} \\
h &= B[\sin(a)] \\
C &= x + y \\
&= (B^2 - B^2\sin^2(a))^{1/2} + (A^2 - B^2\sin^2(a))^{1/2} \\
C &= B[\cos(a)] + (A^2 - B^2 + B^2\cos^2(a))^{1/2}
\end{aligned}
$$

$$C^2 - 2BC[\cos(a)] + B^2[\cos^2(a)] = A^2 - B^2 + B^2[\cos^2(a)]$$

$$\mathbf{C^2 + B^2 - 2BC[\cos(a)] = A^2}$$

**This is the law of cosines.**

To find the area we have:

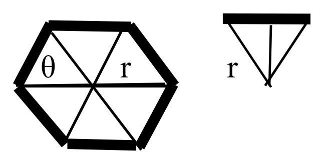
$$
\begin{aligned}
\text{Area} &= \tfrac{1}{2} hC \\
&= \tfrac{1}{2} B \sin(a) C \\
&= \tfrac{1}{2} BC(1 - \cos^2(a))^{1/2} \\
&= \tfrac{1}{2}((BC)^2 - (BC)^2 \cos^2(a))^{1/2} \\
&= \tfrac{1}{2}((BC)^2 - (C^2 + B^2 - A^2)^2 / 4)^{1/2} \\
&= \tfrac{1}{4}((2BC - (C^2 + B^2 - A^2))(2BC + (C^2 + B^2 - A^2)))^{1/2} \\
&= \tfrac{1}{4}((A^2 - (B - C)^2)((B + C)^2 - A^2))^{1/2} \\
&= \tfrac{1}{4}((A + B - C)(A - B + C)(A + B + C)(B + C - A))^{1/2}
\end{aligned}
$$

This equation shows that the sum of the lengths of any two sides of any triangle must be greater than the length of the third side.

## D. Circles from Polygons

Initially we would like to work with regular polygons. Then we can consider polygons with many, many sides so that the polygons approach circles. It is interesting to find the ratio of the circumference to the diameter of these polygons where there can be two lines, each of which can be considered a diameter. Consider the following six-sided polygon:

The angle $\theta$ represents 1/6 of the angular measure of the polygon.




The circumference of the hexagon is $6 \times 2r[\sin(\theta/2)]$, and the inner radius $= r[\cos(\theta/2)]$.

The ratios of the circumference to the two radii are $12[\sin(\theta/2)]$ and $12[\tan(\theta/2)]$ .

If we generalize this, $\theta = 2\pi / n$, $C = n\, 2r \sin \pi/n$, and the ratios are

$(2n)\sin(\pi/n)$ and $(2n)\tan(\pi/n)$.

Let us work with the first ratio and calculate the length of the arc generated by an angle by using the length of one of the n sides $2\sin(\pi/n)$.

Length $= \int r(2\sin(\pi/n)) / 2\pi / n \, d\theta$.

As n becomes very large we can see from the polynomial series for sine that $\sin(\pi/n) / \pi/n$ approaches 1 and Length becomes $\int r \, d\theta$.

Calculus, or continuity, allowed as to use the length of an arc to be the measure of an angle, where the angle is equal to the length of the arc divided by the radius.

## E. Applying the Calculus

Let us calculate the length of a circle using integral calculus. An incremental length is:

$$ds = (dx^2 + dy^2)^{1/2} = (1 + (dy / dx)^2)^{1/2}\ dx$$

The length would be: $\int_{\text{beginning}}^{\text{end}} ds = s$

If the equation of a circle is $x^2 + y^2 = r^2$

then $\quad y = (r^2 - x^2)^{1/2}$

and $\quad dy / dy = -x / (r^2 - x^2)^{1/2}$

$$s = \int_0^r (1 + x^2/(r^2 - x^2))^{1/2}\ dx = \int_0^r r / (r^2 - x^2)^{1/2}\ dx = r \arcsin(x/r)\Big|_0^r = \pi r/2$$

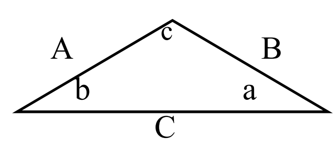
Since this is only one quarter of the circle, the full circumferenc would be 4 x $\pi r/2 = 2\pi r$.

## F. Summary

| | $0^o$ | $90^o$ | $180^o$ | $270^o$ |
|---|---|---|---|---|
| **sin** | 0 | 1 | 0 | -1 |
| **cos** | 1 | 0 | -1 | 0 |
| **tan** | 0 | $\infty$ | 0 | $-\infty$ |

$$\tan(x) = \sin(x) / \cos(x)$$
$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\sin^2(x) + \cos^2(x) = 1$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(\pi/2 - x) = \cos(x)$$
$$\cos(\pi/2 - x) = \sin(x)$$




$$A^2 = C^2 + B^2 - (2BC)\cos(a)$$
$$\text{Area} = \tfrac{1}{4}((a + b - c)(a - b + c)(a + b + c)(b + c - a))^{1/2}$$

## G. Practice

1. Find the area of a triangle whose sides are of lengths 3, 4, and 5 using both the area formula and the base-altitude formula.

2. If the shadow of a pole is 10 feet long and the angle from the end of the shadow to the top of the pole is 60 degrees, what is the height of the pole?

3. If a man six feet tall stands at the seashore, and the radius of the earth is 4000 miles, how far is the horizon?

4. If the distance from a line to the vertex of a triangle is three miles and the length of the end-points of the line are 100 and 200 feet from the perpendicular to that point, what are the angles made with that line and the distant point?